EQUIVALENT SYSTEMS, RESULTANTS OF FORCE AND COUPLE SYSTEM, & FURTHER REDUCTION OF A FORCE AND COUPLE SYSTEM

Today’s Objectives:
Students will be able to:

a) Determine the effect of moving a force.

b) Find an equivalent force-couple system for a system of forces and couples.
What is the resultant effect on the person’s hand when the force is applied in four different ways?
Several forces and a couple moment are acting on this vertical section of an I-beam.

Can you replace them with just one force and one couple moment at point O that will have the same external effect? If yes, how will you do that?
When a number of forces and couple moments are acting on a body, it is easier to understand their overall effect on the body if they are combined into a single force and couple moment having the same external effect.

The two force and couple systems are called **equivalent systems** since they have the same **external** effect on the body.
Moving a force from A to O, when both points are on the vectors’ line of action, does not change the external effect. Hence, a force vector is called a sliding vector. (But the internal effect of the force on the body does depend on where the force is applied).
MOVING A FORCE OFF OF ITS LINE OF ACTION

Moving a force from point A to O (as shown above) requires creating an additional couple moment. Since this new couple moment is a “free” vector, it can be applied at any point P on the body.
RESULTANTS OF A FORCE AND COUPLE SYSTEM
(Section 4.8)

When several forces and couple moments act on a body, you can move each force and its associated couple moment to a common point O.

Now you can add all the forces and couple moments together and find one resultant force-couple moment pair.

\[
\mathbf{F}_R = \sum \mathbf{F} \\
\mathbf{M}_{RO} = \sum \mathbf{M}_c + \sum \mathbf{M}_O
\]
If the force system lies in the x-y plane (the 2-D case), then the reduced equivalent system can be obtained using the following three scalar equations.

\[
\begin{align*}
F_{Rx} &= \Sigma F_x \\
F_{Ry} &= \Sigma F_y \\
M_{RO} &= \Sigma M_c + \Sigma M_O
\end{align*}
\]
EXAMPLE #1

Given: A 2-D force and couple system as shown.

Find: The equivalent resultant force and couple moment acting at A.

Plan:

1) Sum all the x and y components of the forces to find $F_{RA}$.
2) Find and sum all the moments resulting from moving each force to A.
EXAMPLE #1
(continued)

$$+ \rightarrow \Sigma F_{Rx} = 25 + 35 \sin 30^\circ = 42.5 \text{ lb}$$
$$+ \downarrow \Sigma F_{Ry} = 20 + 35 \cos 30^\circ = 50.31 \text{ lb}$$
$$+ \uparrow M_{RA} = 35 \cos 30^\circ (2) + 20(6) - 25(3)$$
$$\quad = 105.6 \text{ lb}\cdot\text{ft}$$

$$F_R = (42.5^2 + 50.31^2)^{1/2} = 65.9 \text{ lb}$$
$$\theta = \tan^{-1} \left( \frac{50.31}{42.5} \right) = 49.8^\circ$$
**EXAMPLE #2**

**Given:** A 2-D force and couple system as shown.

**Find:** The equivalent resultant force and couple moment acting at A.

**Plan:**

1) Sum all the x and y components of the forces to find $F_{RA}$.

2) Find and sum all the moments resulting from moving each force to A and add them to the 500 lb - ft free moment to find the resultant $M_{RA}$. 
EXAMPLE #2 (continued)

Summing the force components:

\[ + \rightarrow \Sigma F_x = \frac{4}{5} 150 \text{ lb} + 50 \text{ lb} \sin 30^\circ = 145 \text{ lb} \]

\[ + \uparrow \Sigma F_y = \frac{3}{5} 150 \text{ lb} + 50 \text{ lb} \cos 30^\circ = 133.3 \text{ lb} \]

Now find the magnitude and direction of the resultant.

\[ F_{RA} = \left( 145^2 + 133.3^2 \right)^{1/2} = 197 \text{ lb} \quad \text{and} \quad \theta = \tan^{-1} \left( \frac{133.3}{145} \right) = 42.6^\circ \]

\[ + \downarrow M_{RA} = \{ (4/5)(150)(2) - 50 \cos 30^\circ (3) + 50 \sin 30^\circ (6) + 500 \} \]

\[ = 760 \text{ lb} \cdot \text{ft} \]
Example #3

**Given:** Handle forces $F_1$ and $F_2$ are applied to the electric drill.

**Find:** An equivalent resultant force and couple moment at point O.

**Plan:**

a) Find $F_{RO} = \Sigma F_i$

Where,

b) Find $M_{RO} = \Sigma M_C + \Sigma (r_i \times F_i)$

$F_i$ are the individual forces in Cartesian vector notation (CVN).

$M_C$ are any free couple moments in CVN (none in this example).

$R_i$ are the position vectors from the point O to any point on the line of action of $F_i$.
SOLUTION

\[ F_1 = \{6\,i - 3\,j - 10\,k\} \text{ N} \]
\[ F_2 = \{0\,i + 2\,j - 4\,k\} \text{ N} \]
\[ F_{RO} = \{6\,i - 1\,j - 14\,k\} \text{ N} \]
\[ r_1 = \{0.15\,i + 0.3\,k\} \text{ m} \]
\[ r_2 = \{-0.25\,j + 0.3\,k\} \text{ m} \]
\[ M_{RO} = r_1 \times F_1 + r_2 \times F_2 \]

\[
M_{RO} = \begin{vmatrix} i & j & k \\ 0.15 & 0 & 0.3 \\ 6 & -3 & -10 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0 & -0.25 & 0.3 \\ 0 & 2 & -4 \end{vmatrix} \text{ N} \cdot \text{m}
\]
\[= \{0.9\,i + 3.3\,j - 0.45\,k + 0.4\,i + 0\,j + 0\,k\} \text{ N} \cdot \text{m} \]
\[= \{1.3\,i + 3.3\,j - 0.45\,k\} \text{ N} \cdot \text{m} \]
FURTHER REDUCTION OF A FORCE AND COUPLE SYSTEM
(Section 4.9)

If \( F_R \) and \( M_{RO} \) are perpendicular to each other, then the system can be further reduced to a single force, \( F_R \), by simply moving \( F_R \) from O to P.

In three special cases, concurrent, coplanar, and parallel systems of forces, the system can always be reduced to a single force as long as \( F_R \neq 0 \).
Concurrent means lines of action of all the forces pass through the same point.

By definition, $M = R \times F_R$ will be perpendicular to $F_R$. 
Coplanar – all forces in same plane
– effectively a 2D problem
– $F_R$ also in plane

For a 2D problem, all moments are out of or into page, so $M_O \perp F_R$
Parallel forces can create no moments in their own direction.

Hence the only moments are perpendicular to the forces as is required.
EXAMPLE #4 - Coplanar

Given: A 2-D force and couple system as shown.

Find: The equivalent resultant force and couple moment acting at A and then the equivalent single force location along the beam AB.

Plan:

1) Sum all the x and y components of the forces to find $F_{RA}$.
2) Find and sum all the moments resulting from moving each force to A.
3) Shift the $F_{RA}$ to a distance $d$ such that $d = M_{RA}/F_{Ry}$
EXAMPLE #4
(continued)

\[ \sum F_{Rx} = 25 + 35 \sin 30^\circ = 42.5 \text{ lb} \]
\[ \sum F_{Ry} = 20 + 35 \cos 30^\circ = 50.31 \text{ lb} \]
\[ M_{RA} = 35 \cos 30^\circ (2) + 20(6) - 25(3) = 105.6 \text{ lb} \cdot \text{ft} \]

\[ F_R = \left( (42.5^2 + 50.31^2) \right)^{1/2} = 65.9 \text{ lb} \]
\[ \theta = \tan^{-1}\left( \frac{50.31}{42.5} \right) = 49.8^\circ \]

The equivalent single force $F_R$ can be located on the beam AB at a distance $d$ measured from A.

\[ d = \frac{M_{RA}}{F_{Ry}} = \frac{105.6}{50.31} = 2.10 \text{ ft}. \]
EXAMPLE #5 Parallel

**Given:** The building slab has four columns. \( F_1 \) and \( F_2 = 0 \).

**Find:** The equivalent resultant force and couple moment at the origin \( O \). Also find the location \((x,y)\) of the single equivalent resultant force.

**Plan:**

1) Find \( F_{RO} = \sum F_i = F_{RzO} k \)

2) Find \( M_{RO} = \sum (r_i \times F_i) = M_{RxO} i + M_{RyO} j \)

3) The location of the single equivalent resultant force is given as \( x = -\frac{M_{RyO}}{F_{RzO}} \) and \( y = \frac{M_{RxO}}{F_{RzO}} \)
EXAMPLE #5
(continued)

\[ F_{RO} = \{-50 \, k - 20 \, k\} = \{-70 \, k\} \text{ kN} \]

\[ M_{RO} = (10 \, i) \times (-20 \, k) + (4 \, i + 3 \, j) \times (-50 \, k) \]

\[ = \{200 \, j + 200 \, j - 150 \, i\} \text{ kN} \cdot \text{m} \]

\[ = \{-150 \, i + 400 \, j\} \text{ kN} \cdot \text{m} \]

The location of the single equivalent resultant force is given as,

\[ x = -\frac{M_{Ryo}}{F_{Rzo}} = -\frac{400}{-70} = 5.71 \text{ m} \]

\[ y = \frac{M_{Rxo}}{F_{Rzo}} = \frac{-150}{-70} = 2.14 \text{ m} \]
EXAMPLE #6 Concurrent

**Given:** The block is acted upon by three forces.

**Find:** The equivalent resultant force and couple moment at the origin O. Also find the location \((0, y, z)\) of the single equivalent resultant force.

**Plan:**

1) Find \(F_{RO} = \sum F_i\)

2) Find \(M_{RO} = \sum (r_i \times F_i) = r \times F_{RO}\)

3) The location of the single equivalent resultant force is given as \(x = M_{RyO}/F_{RzO}\) and \(y = M_{RxO}/F_{RzO}\)
EXAMPLE #6 Continued

\[ F_3 = 70 \text{ N} \]

\[ F_1 = 14 \text{ N} \]

\[ F_2 = 48 \text{ j N} \]

\[ F_3 = (70 \text{ N})(-2i - 6j + 3k)/7 \]

\[ = (-20i - 60j + 30k) \text{ N} \]

\[ F_R = (-20i - 12j + 16k) \text{ N} \]

\[ M_O = r \times F_R = \begin{vmatrix} i & j & k \\ 2 & 6 & 0 \\ -20 & -12 & 16 \end{vmatrix} \text{ m·N} \]

\[ = (6*16)i - j(2*16) + k(-24+120) \text{ m·N} \]

\[ = \{96i - 32j + 96k\} \text{ m·N} \]
EXAMPLE #6 Continued

Want

\[ M_O = \mathbf{r}' \times \mathbf{F}_R = \begin{vmatrix} i & j & k \\ 0 & y & z \\ -20 & -12 & 16 \end{vmatrix} \quad m \cdot \mathbf{N} = \{96i - 32j + 96k\} \quad m \cdot \mathbf{N} \]

Or

\[(16y + 20z)i - j(20z) + k(20y) = 96i - 32j + 96k\]

Get \( z = \frac{32}{20} = 1.6, \quad y = \frac{96}{20} = 4.8 \)

Check: \( 16 \times 4.8 + 20 \times (1.6) = 96 \) ✔

If \( y \) and \( z \) were not on the face, solution would be unphysical.
**Wrench or Screw**

In general, $F_R$ is not $\perp M_O$ but can still simplify somewhat.

$$M_O = M_{\parallel} + M_{\perp}$$

Can find a location, such that $F_R$ and $M_{\perp}$ have the same external effect.

Fig. 4–43
Approach

• Find $F_R$ and $M_O$ as usual

• Use dot product to find $M_\parallel$

• Recall $M_\parallel = (M_O \cdot u_F) \, u_F$

• Next $M_\perp = M_O - M_\parallel$

• Find $r$ such that $r \times F_R = M_\perp$
EXAMPLE #7

**Given:** The block is acted upon by three forces.

**Find:** The equivalent resultant force and couple moment at the origin O. Also find the location \((0, y, z)\) of the wrench.

\[
\begin{align*}
F_1 &= 25 \text{ N} \\
F_2 &= 48 \text{ N} \\
F_3 &= 70 \text{ N}
\end{align*}
\]

**Plan:**

1) Find \(F_{RO} = \sum F_i\)

2) Find \(M_{RO} = \sum (r_i \times F_i)\)

3) Break \(M_{RO}\) into components

4) Find \(r \times F_{RO} = M_\perp\)

5) Wrench is \(F_{RO}, M_\parallel, \) and location \(r\)
EXAMPLE #7 (continued)

\[ F_3 = 70 \text{ N} \]

\[ F_1 = 25 \text{ N} \]

\[ F_2 = 48 \text{ N} \]

\[ F_1 = -25 \text{ k N} \]

\[ F_2 = 48 \text{ j N} \]

\[ F_3 = (70 \text{ N})(-2i - 6j + 3k)/7 \]

\[ = (-20i - 60j + 30k) \text{ N} \]

\[ F_R = (-20i - 12j + 5k) \text{ N} \]

\[
M_O = \begin{vmatrix} i & j & k \\ 0 & 6 & 0 \\ 0 & 0 & -25 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 0 & 48 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & 3 \\ -20 & -60 & 30 \end{vmatrix} \quad \text{m \cdot N}
\]

\[ = (-150)i + (96)k + (180i - 60j) \quad \text{m \cdot N} \]

\[ = \{30i - 60j + 96k\} \quad \text{m \cdot N} \]
EXAMPLE #7 (continued)

Since $F_R = (-20i - 12j + 5k) \text{ N}$, $u_F = (-20i - 12j + 5k)/\sqrt{569}$

$$M_{\parallel} = (M_O \cdot u_F) \cdot u_F$$

$$= \{30i - 60j + 96k\} \cdot (-20i - 12j + 5k)/\sqrt{569}$$

$$\cdot (-20i - 12j + 5k)/\sqrt{569} \text{ m.N}$$

$$= (-600 + 720 + 480)(-20i - 12j + 5k)/569 \text{ m.N}$$

$$= 600 (-20i - 12j + 5k)/569 \text{ m.N}$$

$$= i (-21.0896) + j (-12.6538) + k (5.2724) \text{ m.N}$$

$$M_{\perp} = M_O - M_{\parallel}$$

$$= \{30i - 60j + 96k\} - \{-21.0896i - 12.6538j + 5.2724k\}$$

$$= \{51.0896i - 47.3462j + 90.7276k\} \text{ m.N}$$
EXAMPLE #7 Continued

Want \( r \times F_R = M_\perp \)

\[
\begin{vmatrix}
  i & j & k \\
  0 & y & z \\
-20 & -12 & 5
\end{vmatrix} = \{51.0896 \, i - 47.3462 \, j + 90.7276 \, k\}
\]

Or

\[(5y+12z)i - j(20z) + k(20y) = 51.0896 \, i - 47.3462 \, j + 90.7276 \, k\]

Get \( z = 47.3462/20 = 2.3673 \), \( y = 90.7276/20 = 4.5364 \)

Check: \( 5*4.5364 + 12*(2.3673) = 51.0896 \) ✔

If \( y \) and \( z \) were not on the face, solution would be unphysical.
1. The forces on the pole can be reduced to a single force and a single moment at point ____.
   1) P  2) Q  3) R  4) S  5) Any of these points.

2. Consider **two couples** acting on a body. The simplest possible equivalent system at any arbitrary point on the body will have
   1) one force and one couple moment.
   2) one force.
   3) one couple moment.
   4) two couple moments.
ATTENTION QUIZ

1. For this force system, the equivalent system at P is
   ____________

A) \( F_{RP} = 40 \text{ lb (along +x-dir.) and } M_{RP} = +60 \text{ ft } \cdot \text{ lb} \)
B) \( F_{RP} = 0 \text{ lb and } M_{RP} = +30 \text{ ft } \cdot \text{ lb} \)
C) \( F_{RP} = 30 \text{ lb (along +y-dir.) and } M_{RP} = -30 \text{ ft } \cdot \text{ lb} \)
D) \( F_{RP} = 40 \text{ lb (along +x-dir.) and } M_{RP} = +30 \text{ ft } \cdot \text{ lb} \)
ATTENTION QUIZ

2. Consider three couples acting on a body. Equivalent systems will be _______ at different points on the body.

A) different when located

B) the same even when located

C) zero when located

D) None of the above.